Permutations

Consider a volleyball tournament with 8 different teams: Esquimalt, Lambrick, Mt. Doug, Oak Bay, Quadra Elementary, Reynolds, Spectrum, and Vic High. To save some typing I will refer to them by their initial letter: E, L, M, O, Q, R, S, and V. They were written in alphabetical order but in sports we are usually interested in putting teams in order from first to last, depending on how they played. The question often arises as to how many distinct ways can those teams finish, or, to put it another way, how many possible finishing orders, or permutations, can there be for those 8 teams?

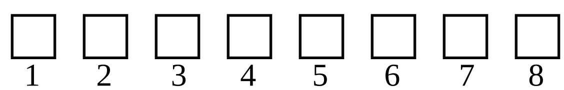
If there were only two teams this problem would be easy. Consider Reynolds and Quadra. The two possible finishes are (from first to last) RQ and QR.

With three teams it’s still easy. Consider Esquimalt, Reynolds, and Quadra. The possible finishes are: ERQ, EQR, REQ, RQE, QER, QRE. I listed them systematically, trying to preserve alphabetical order until I had to reverse it to write one of the possible outcomes.

With four or more teams it gets snarlier, because there are more possible *permutations*. (By now you should realize that the word “permutation” means, loosely, “a list of items in which the order of the items matters, i.e. two or more lists that have the same items but in a different order, would be considered two separate permutations.) It becomes easier to miss a listing or to write one out twice, even if you are very careful and follow some logical plan.

Luckily for us there is an easier, logical way to approach the problem.

Think of the problem as being one of stuffing letters into boxes, one letter per box. We have 8 letters, E, L, M, O, Q, R, S, and V. We have 8 boxes. Each box represents a finishing place.



Now think... how many boxes are available for the E, before we start the stuffing? Yes, there are 8 boxes available! So we put the E into one of the boxes, it doesn’t matter which one. Now how many are left for the L? Right - there are 7 boxes left! No matter which box the E went into, there are still 7 other boxes for the L. Now, for every one of the locations the E could have gone into, there are 7 locations for the L. That means there must be 8 x 7, or 56 ways to stuff the boxes with E and L.

Now let’s say E and L are sitting in whatever boxes we put them in. How many boxes are left to be a little home for the M? Right - six boxes! For every variation of the E and L (there were 8 x 7, or 56 variations) there are 6 possible boxes left for the M! That means there must be 56 x 6, or 336 ways to stuff the boxes with E, L and M!

Let’s do this one more time - isn’t it fun! Let’s say E and L and M are sitting in whatever boxes we put them in. How many boxes are available for O? Right - 5 boxes! For every variation of the E, L and M (there were 8 x 7 x 6, or 336 variations) there are 5 possible boxes left for the O! That means there must be 336 x 5, or 1680 ways to stuff the boxes with E, L, M, and O. It appears that Elmo can get stuffed 1680 ways!

Some of you may be thinking, “Wait a minute! This was about the finishing place in a volleyball tournament. If we looked at every possible finishing place for E right at the start, then how does L or M or O get to possibly finish in the same spot. If we put E into first place, that means the others can’t have it!” Well, fear not, friends of Lambrick, Mt. Doug, or Oak Bay, just remember that we are not looking at actual results here - we are looking at all *possible* results (and yes, Quadra could come in first - have you seen the size of those kids???!!!) So we have actually included all the situations where L, M, and O could come in first - they all would happen in the variations where E doesn't come in first.

By now you have probably seen the way this process is heading. If there are 8 different things to be put in 8 different places, then there are 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 distinctly different ways to do it! That's 40320 ways, or 40320 permutations for 8 items taken 8 at a time.

Important sidebar:

Some of you may recognize the structure: 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1

It is called "eight factorial" and is abbreviated in mathematics as "8!"

Similarly 5! means 5 x 4 x 3 x 2 x 1

Most scientific calculators have a "factorial" key. It's the "x!" one, and it is usually a second function key.

So if there were 5 teams, there would be 5!, or 5 x 4 x 3 x 2 x 1, or 120 ways the tournament could finish.

Summary: In general, if there are "n" different items to be considered "n" at a time, the number of permutations will be "n!"

Now, suppose you don't care how many possible finishes there are for all the teams. You only want to know how many permutations there are for first, second, and third place, in our eight-team tournament. You want to know how many permutations there can be in a group of three items, given that there are eight items to pick from. Remember our boxes? This solution requires a slight variation on the box theme. Think of it this way.

How many teams could finish in first place? Right - eight teams! There are 8 possible first place finishers. That leaves only seven possible second place finishers! For each team that could finish in first place, there are seven other teams left who could finish in second. That gives a total of 8 x 7, or 56 ways for two teams out of eight to finish first and second!

Now suppose the first and second place teams are decided. That leaves only six possible third place finishers!. For each permutation of first-and-second place finishers, there are six teams who could finish third. That gives a total of 56 x 6, or 8 x 7 x 6, or 336 possible permutations of three teams out of eight finishing first, second, and third.

Of course we all know that this mathematics will be spoiled in reality because Reynolds will always finish first, leaving only 7 other spots!

But on the off-chance that Reynolds doesn't win, can we see how to generalize this idea for any number of teams and any number of finishing spots? Sure.

The basic idea is to think of the factorial expression 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 and imagine eliminating the last five terms.

In our box metaphor, it goes something like this:

□	□	□	□	□	□	□	□
1	2	3	4	5	6	7	8

In box 1, there were 8 possible teams. Once one team is fixed there it leaves 7 possible teams for box 2. Then 6 possible teams for box 3, and so on. However many finishing places you want to consider, that's precisely the number of terms you have to multiply from the left side of the factorial expansion 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1. The rest don't count - they get "eliminated". To "eliminate" terms from a string of multiplied factors, you simply divide! The common factors "cancel" out. Observe:

$$\frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 8 \times 7 \times 6$$

$$= 336$$

Now the denominator above is 5 x 4 x 3 x 2 x 1, or 5!

Another way to say 5! is (8 - 3)! Note the relevance of the 8 and the 3. The 8 is the number of teams and the 3 is the number of finishing places. We are not interested in the lowest five finishing places so we eliminated them.

So a more compact expression for our permutations-of-the-top-three-finishers would be:

$$\frac{8!}{(8-3)!} = 336$$

Summary: In general, for "n" different items, taken "r" at a time, the number of permutations is:

$$\frac{n!}{(n-r)!}$$

What you have just been through is a development of the general expression for "permutations".

(Note: we didn't cover the case where some of the items are the same, like the list of letters AAABBCDE. Check your text to see how that situation is handled.)

The companion concept for permutations is "combinations". Let's head there now! It's as easy as reading the next line.

Combinations

Now imagine a California psychologist has just convinced the school board that ranking teams in a tournament does irreparable harm to student's self-esteem.

(Ques.: What goes, "I think I can, I think I can, I think I can, I think I can, I think I can, I think I can, I can, I can, I can, I can, I can, I can, I can!"

Ans.: A counsellor working up a head of self-esteem.)

The school board partially agreed and decided that the same prize would be given to all the top three teams, but there would be no ranking as first, second, and third. They would all just be called, "The Top Three". In other words, the order of those top three didn't matter.

In that case, how many *combinations* of three teams taken from a pool of eight can there be?

(I hope you have figured out that the word "combination" means, loosely, a list of items taken from a longer list, where the order of items in the shorter list does not matter.)

Again, it is relatively easy to work out a simple case. For example, consider the list of three letters, "abc". There are six *permutations* of this list when taken two at a time, namely *ab*, *ac*, *ba*, *bc*, *ca*, and *cb*. But if we ask how many *combinations* of two letters can we make, there are only three, because if the order of the letters doesn't matter, then *ab* is the same combination as *ba*, *ac* the same as *ca*, and *bc* the same as *cb*.

For our particular team problem, consider the number of permutations of top three finishers, which was 336. This is six times the number of *combinations* possible. How so?

Let's consider the permutation, RMS. If the order of these letters didn't matter, then this finish would be the equivalent of RSM, MRS, MSR, SMR, and SRM; in other words, all the permutations of the list "RMS" itself!. These six *permutations* amount to only one *combination* of the three letters. Now there were other possible finishes with different teams from the original eight. The possible finish involving R and M and O is one combination only, but has six permutations too. Every one of the combinations of three teams from the list of eight has 6 permutations. If there were 336 permutations overall, then there must be only one-sixth as many combinations. That would be 56 combinations.

Now, by our previous expression for the number of permutations possible for a list of "n" different items, namely "n!", we see that if we have three different items, we would have 3 x 2 x 1, or 6 permutations. In our revised problem , we are looking for the number of *combinations* of eight things taken three at a time. We can arrive at a formula for this if we simply divide the number of permutations of eight things taken three at a time, by the number of permutations of three things period! We get the following:

$$\frac{8!}{3!(8-3)!} = 56$$

Note the difference between this calculation and the permutation calculation. There is an 3! at the beginning of the denominator. This represents the number of permutations of any one of the possible combinations of top three finishers.

Summary: In general, for "n" different items, taken "r" at a time, the number of combinations is:

$$\frac{n!}{r!(n-r)!}$$

Wouldn't it be nice if there was a button you could push on your calculator that would just tell you how many combinations or permutations there are in any situation? Well, if you have a certain kind of scientific calculator, there is! Or rather, there are! You usually need to push two buttons, and they are usually "second function" buttons.

Here's an example. Let's find the number of *combinations* of eight things taken three at a time. Push the "8", then on most calculators you push a button with an "x" and a "y" and a couple of arrows on it, then push the "3" and then the "nCr" button (remember they may be "second function" buttons). You will get "56". And to find the number of *permutations* of eight things taken three at a time. Push the "8", then on most calculators you push a button with an "x" and a "y" and a couple of arrows on it, then push the "3" and then the "nPr" button (remember they may be "second function" buttons). You will get "336". See the teacher if you are confused.